A General Algorithm of the Boundary Integral Method for Solving Laplace's Mixed Boundary Value Problem

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Abstract
When greater precision is required, boundary elements have become a potent substitute for finite elements. The fact that boundary elements only require the surface to be descretized—rather than the volume—is one of its most crucial aspects. Here, elliptic partial differential equations can be solved using a generalised boundary integral technique methodology. A problem of practical interest providing the solution of the Laplace equation for potential flow with mixed boundary problems serves as an illustration of the approach's extensive applicability. Tables and figures present the results and patterns, which are found to be in good agreement when compared to Brebbia [1].

Keywords: Laplace equation, boundary elements, and boundary integral approach.

Introduction

In many engineering and physical scientific applications, the finite element method has been shown to be insufficient or inefficient, and in certain situations, it can be quite difficult to apply. Still, finite element analysis is a a somewhat drawn-out procedure because meshes in the piece or domain under consideration must be defined or redefined. The boundary element technique, which has shown to be an effective substitute for finite elements, was developed with this goal in mind [2, 3].

Boundary elements have become a potent substitute for finite elements, especially when greater accuracy is needed because of issues like stress concentration or infinite domains. The fact that boundary elements only require the surface to be descretized—rather than the volume—is one of its most crucial aspects. Boundary element approaches are therefore simpler to apply.

It is possible to get the fundamental boundary integral equations needed for this method by taking weighted residuals into account. Moreover, it can be inferred using Green's third identity principle. Boundary element approaches gained the interest of scientists and engineers in the 1980s. Brebbia [4] has thoroughly compared this method to finite difference and finite element approaches and discussed how beneficial it is in comparison to those methods.

• Review of Litrature

The creation of Boundary Elements Methods is credited to Brebbia and his associates. In Brebbia and Dominguez's work (refer to [1] and [5]), the term "Boundary Elements Methods" was originated. These publications presented a comparison between Finite Elements and Boundary Elements methods. Additionally, they offered methods for applying Boundary Elements Methods to possible issues. Potential flow issues are widely applicable in the fields of aerodynamics, engineering sciences, and mathematics. Laplace's equation, a significant elliptic partial differential equation, generally governs them. Even so, a few publications about the use of "integral equation methods," particularly for solid mechanics problems like "torsion problems," have already been published. But because of their restricted applicability, they were unable to get much attention. Brebbia and Ferrante's work [6] is credited with bringing this technique to light. Brebbia and Wandland also provide a detailed analysis [7].

The method is used on a Laplace boundary value issue in this instance with mixed boundary conditions, or boundary conditions that include both Dirichlet and Neumann types. But the method can also be applied to solve other kinds of elliptic partial differential equations, such as Poisson ones.

• Algorithms for BEM

Using basic equations, an algorithm has been created to compute the results by minimising the residuals and adding all significant elements in accordance with the following standard process. Xianyun Qin et al. [8] have provided an outline for singular integrals on 3D boundary elements.

• Basic integral equation

In a domain (two or three dimensional) , the Laplace equation is

2u=o}in ~(1)

with the further requirements on the ?? border

"Essential" Circumstances of the form u= u on ??1 • •"Natural" Requirements like q = u/n = q on ??2 where the bar denotes that these values are known and n is the normal to the boundary?? =??1 +??2.

Theoretically, by orthogonalizing the values of u and q with respect to a weighted function u* and obtaining

derivatives on the boundary q* = わ u*/ザ n, one can minimise the error generated in the above equation if the actual (but unknown) values of u and q are substituted by an estimated solution.
Stated differently, if R represents the residuals, then R = 2u ㉫o can be written generally.

R1 = u - u ₹ 0 (3)
R2 = q - q o

where the numbers u and q are approximations. (The argument's applicability is not diminished by the possibility that one or more of the residuals are exactly zero.)

It is now possible to perform the weighting as indicated below.

Two authors have developed the singular function boundary integral approach for elliptic problems: Xenophontos [9] and Christodoulou et al. [10].

- **Fundamental solution**

The field created by a concentrated unit charge operating at a location "i" is represented by the fundamental solution, u*, which also satisfies Laplace's equation. This charge propagates from i to infinity without taking boundary circumstances into account. This enables the solution to be written.

2 u* + i = 0 (8)

where い i is a Dirac Delta function that is equal to zero everywhere else and tends to infinity at x=xi. On the other hand, the integral of i equals one.

Any other function multiplied by the integral of the Dirac delta function equals the value of that function at the point xi. Therefore

- **Boundary integral equation**

We have now deduced an equation (10) which is valid for any point within the Ω domain. In boundary elements it is usually preferable for computational reasons to apply equation (10) on the boundary and hence we need to find out what happens when the point x^i is on Γ. A simple way to do this is to consider that the point i is on the boundary but the domain itself is augmented by a hemisphere of radius ε (in 3D) as shown in Fig. 1 (for two D the same applies but we will consider a semicircle instead). The point x^i is considered to be at the centre and then the radius ε is taken to zero. The point will then become a boundary point and the resulting expression the specialization of (10) for a point on Γ.

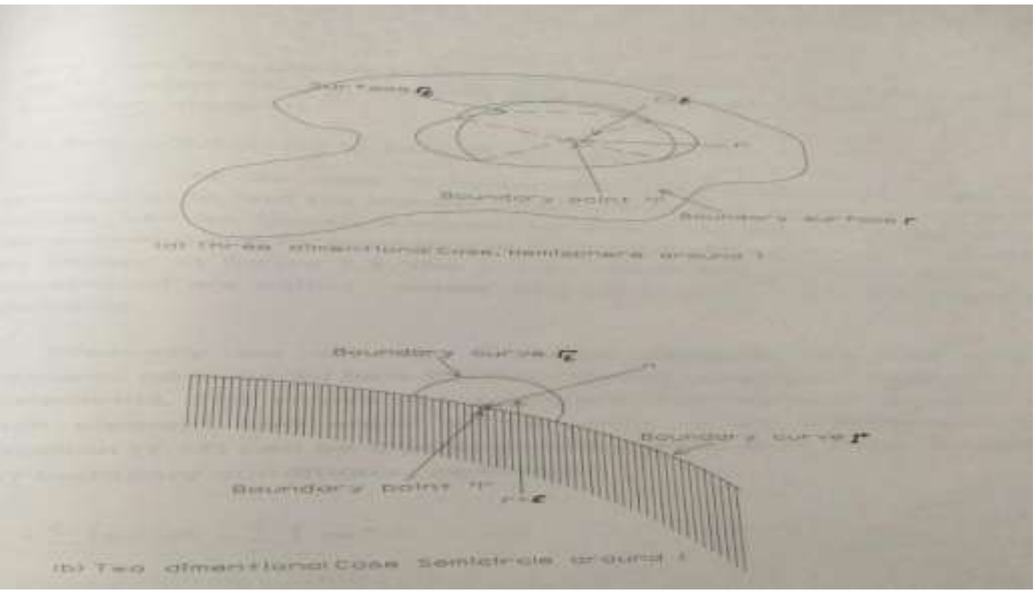


Fig. 1. Boundary points for two and three dimensional case

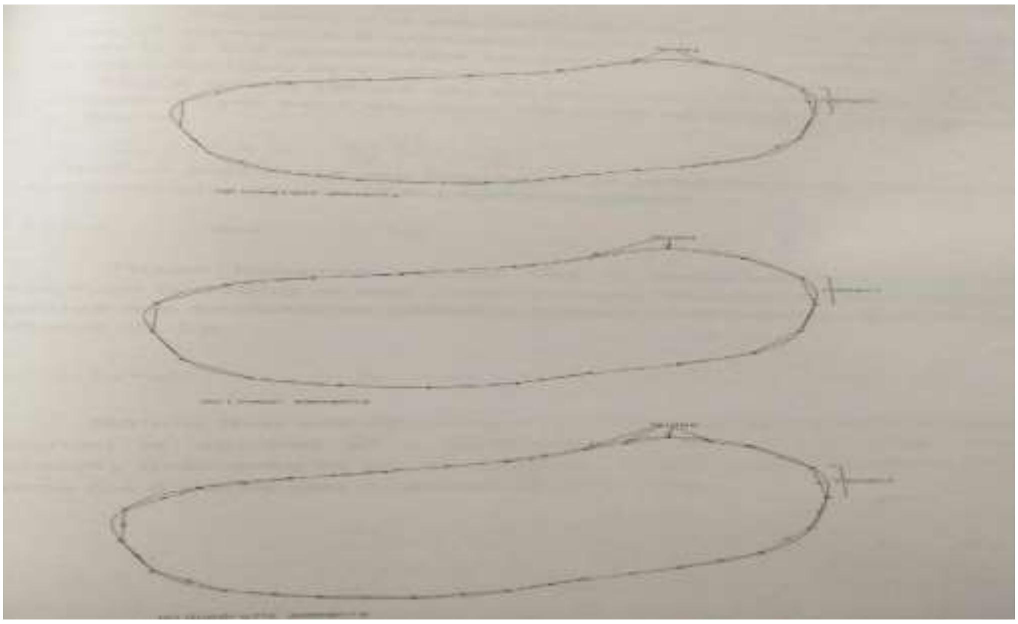


Fig. 2. Different types of boundary elements

It is important at this stage to differentiate between two types of boundary integrals in (10) as the fundamental solution and its derivative behave differently. Consider for the sake of simplicity equation (10) before any boundary conditions have been applied, i.e.

$$u^{i} + \int_{\Gamma} uq^{*}\, d\Gamma = \int_{\Gamma} u^{*} q\, d\Gamma \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

Here $\Gamma = \Gamma_1 + \Gamma_2$ and satisfaction of the boundary conditions will be left for the latter on. Integrals of the type shown on the right hand side of (13) are easy to deal with as they present a lower order singularity, i.e. for three dimensional cases the integral around Γ_ε gives:

$$\lim_{\varepsilon \to 0} \left\{ \int_{\Gamma} qu^{*}\, d\Gamma \right\} = \lim_{\varepsilon \to 0} \left\{ \int_{\Gamma} q \frac{1}{4\pi\varepsilon} d\Gamma \right\} = \lim_{\varepsilon \to 0} \left\{ q \frac{2\pi\varepsilon^{2}}{4\pi\varepsilon} \right\} = .0 \ldots\ldots$$

In other words nothing occurs to the right hand side integral when (10) or (13) are taken to the boundary . The left hand side integral however behaves in a different manner . Here we have around Γ_ε the following result,

where the integrals are in the sense of cauchy principal value. This is the boundary integral equation generally used as a starting point for boundary elements.

- **The boundary element methods**

Now let's look at how to discretize expression (17) in order to determine the system of equations that yields the boundary values. For the sake of simplicity, let us assume that the body is two-dimensional, and that its boundary is composed of N segments, or elements, as Fig. 2 illustrates. "Nodes" are the spots that are assumed to be in the midst of the elements and are where the unknown values are taken into account. AL-Jawary [11] provides a thorough description of the numerical solution of the diffusion problem in two dimensions with variable coefficients.

② BEM for ever-present elements

The boundary is taken to be partitioned into N elements for the constant elements that are under consideration. It is expected that u and q always have the same values at the mid-element node and are constant across all elements. Before imposing any boundary constraints, equation (17) can be discretized for a given point "i" in the manner shown below:

If we now consider that the position of i can also change from 1 to N, that is, that the fundamental solution is applied at each node consecutively, then applying (21) to each boundary point in turn results in a system of equations.
Now let's make a call.

When Ij, hij
Hij + 1/2 = (22) Hij when i = j

Equation (21) can therefore now be expressed as

where U and Q are N-length vectors and H and G are NxN matrices.

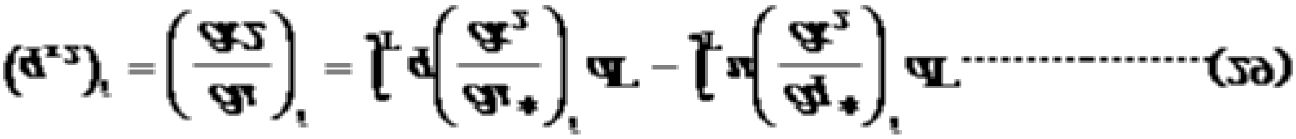
It can be observed that N1 values of u and N2 values of q are known on □1 and □2, respectively (□1+□2 = □), meaning that the system of equations only has N unknowns (24). In order to include these boundary conditions in equation (24)

to reorganise the system by shifting the H and G columns from side to side. One can write, after all unknowns have been transferred to the left,

F^(25) = AXE

where X is a vector representing the boundary values of the unknowns u and q. The corresponding columns are multiplied by the known values of u or q to find F. Interestingly, instead of only the potential as in finite elements, the unknowns now consist of a blend of the potential and its derivative. This results from the boundary element's "mixed" formulation and offers the method a significant advantage over finite elements.

Now that equation (25) has been solved, all of the boundary values are known. After doing this, any internal value of u or its derivatives can be computed. Formula (10) can be expressed as follows: The values of u's are computed at any internal point 'i' using the formula:



Notice that the derivatives are carried out only on the fundamental solution functions u* and q* as we are computing the variations of the flux around the 'i' point.

Computation of integrals for internal points in (27), (28) and (29) are usually carried out numerically.

- **BEM for linear elements**

Here we consider linear elements instead of constant elements. We consider a linear variation of u and q for which case the nodes are considered to to be at the ends of the elements as shown in Fig. 3.

The integral equation (13) for linear elements is written as

Solving exactly as in the case of constant elements, we get values of unknowns.

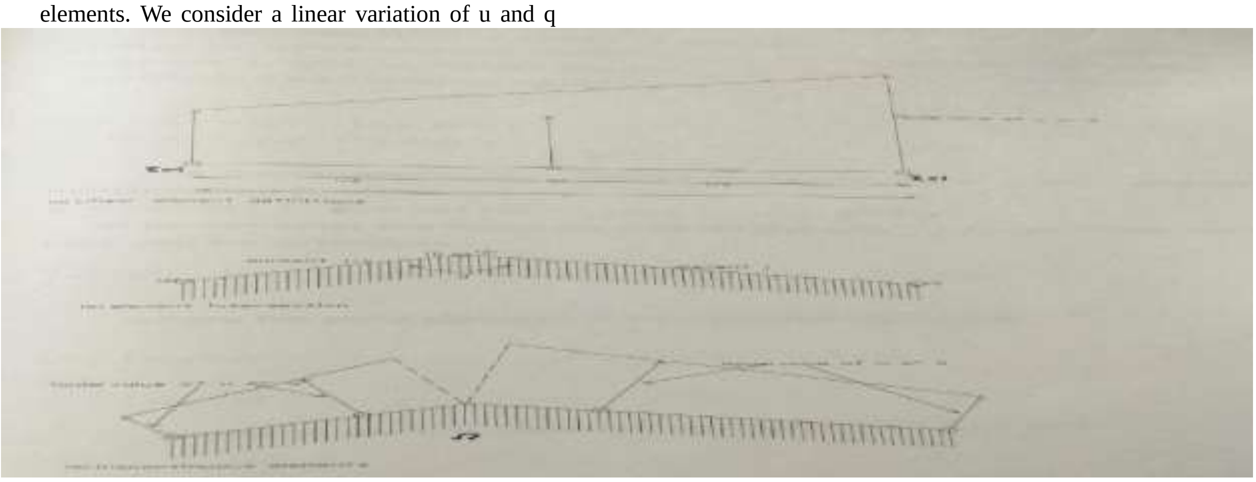


Fig. 3. Linear elements basic definitions and corner treatment

• The Problem:

Consider the Laplace's equation

$$\nabla^2 u = 0 \tag{42}$$

in a domain Ω which is a square with each side unity as shown in Fig. 4

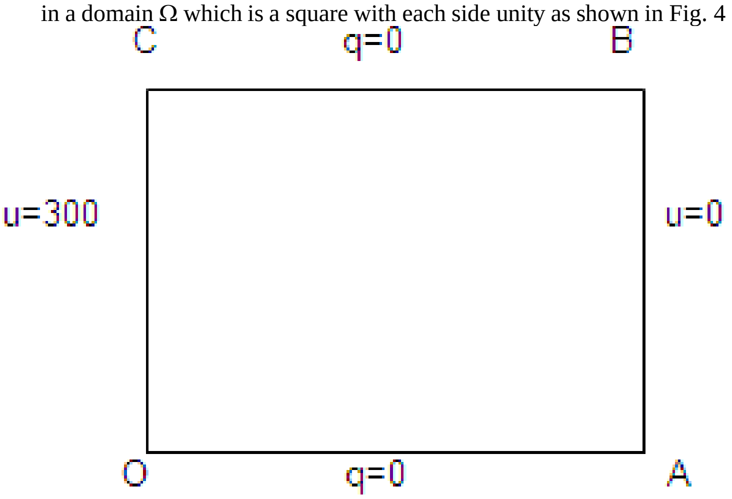


Fig. 4. Domain Ω

Boundary conditions are mixed type as follows u =

300 along OC

u = 0 along AB (43)

q = 0 along OA and BC

• Results and Discussion

The numerical results are obtained using the given algorithm for constant elements by modern computing device and techniques. We discretize the domain into boundary elements and internal nodes as follows:

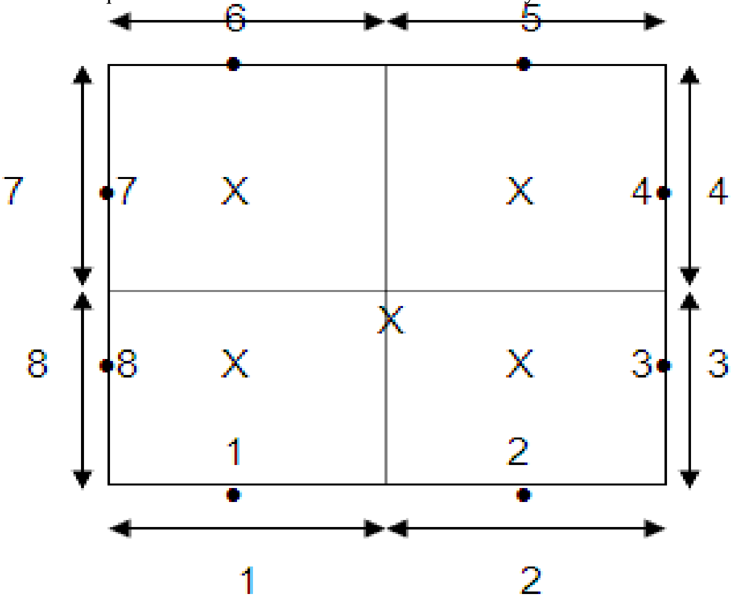


Fig. 5. X →Internal points

Here the domain is descretized into 8 boundary elements and 5 internal nodes. Computational results are depicted in Table 1 for boundary nodes and in Table 2 internal nodes. Pattern of results in Table 1 is in good agreement with given boundary conditions. Moreover, values of potential at internal nodes from Table 2 are shown in Fig. 6

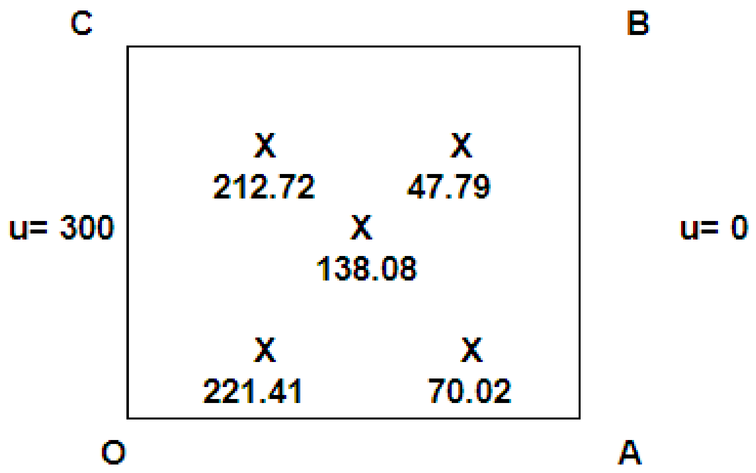


Fig. 6. Values of potential at 5 internal nodes

Above computational results also validate the given boundary conditions, since value of u is zero along AB and 300 along OC.

The same problem is recomputed by taking 16 boundary elements and nine internal nodes as follows:

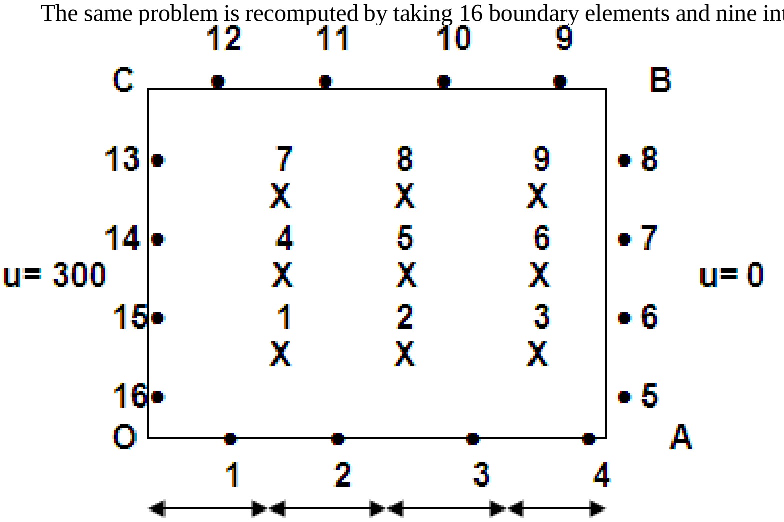


Fig. 7. X Internal nodes

Computational results are given in Table 3 for boundary nodes and in Table 4 for internal nodes. Values of potential u at internal nodes are shown in Fig. 8 here.

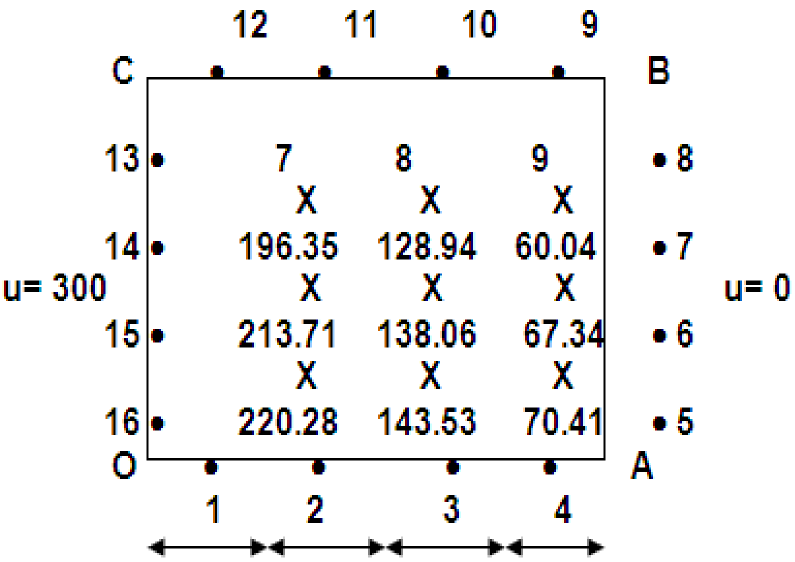


Fig. 8. Values of potential at 9 internal nodes

Results are in excellent agreement with given boundary conditions as well as with results obtained by 8 elements and 5 internal nodes.

From Tables 1-4, it is apparent that results using constant elements have same pattern and features. The same conclusion was expected by Brebbia [1]. Pattern of computational results at lower boundary OA is shown in Fig. 9 for constant elements.

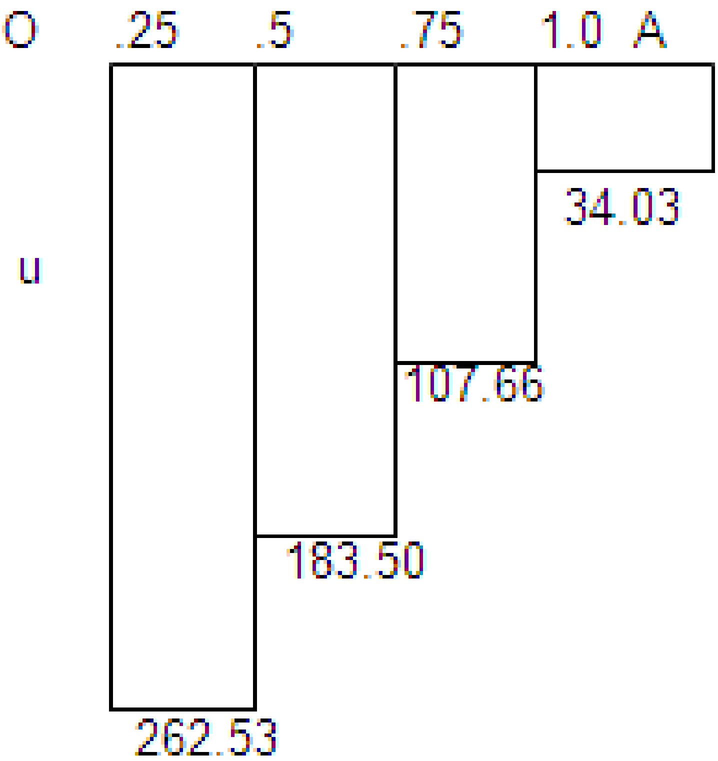


Fig. 9.

Table 1. Value of potential and potential derivatives taking 8 Bounday nodes

Boundary nodes	X	Y	Potential	Potential derivative
1	0.25	0.00	224.62	0.000
2	0.75	0.00	69.284	0.000
3	1.00	0.25	0.000	-303.36
4	1.00	0.75	0.000	-129.40
5	0.75	1.00	0.000	-275.76
6	0.25	1.00	210.67	0.000
7	0.00	0.75	300.00	373.80
8	0.00	0.25	300.00	334.93

Table 2. Values of potential taking 5 internal nodes

Internal nodes	X	Y	Potential
1	0.25	0.25	221.41
2	0.75	0.25	70.02
3	0.50	0.50	138.08
4	0.25	0.75	212.72
5	0.75	0.75	47.79

Table 3. Value of potential and potential derivatives taking 16 boundary nodes

Boundary nodes	X	Y	Potential	Potential derivative
1	0.125	0.000	262.53	0.00
2	0.375	0.000	183.50	0.00
3	0.625	0.000	107.66	0.00
4	0.875	0.000	340.36	0.00
5	1.000	0.125	0.00	-301.48
6	1.000	0.375	0.00	-271.77
7	1.000	0.625	0.00	-254.92
8	1.000	0.875	0.00	-114.41
9	0.875	1.000	0.00	-231.65
10	0.625	1.000	86.68	0.00
11	0.375	1.000	155.93	0.00
12	0.125	1.000	205.42	0.00
13	0.000	0.875	220.48	0.00
14	0.000	0.625	300.00	506.37
15	0.000	0.375	300.00	322.48
16	0.000	0.125	300.00	334.92

Table 4. Values of potential taking 9 internal nodes

Internal nodes	X	Y	Potential
1	0.25	0.25	220.28
2	0.50	0.25	143.53
3	0.75	0.25	70.41
4	0.25	0.50	213.71
5	0.50	0.50	138.06
6	0.75	0.50	67.35
7	0.25	0.75	196.35
8	0.50	0.75	128.94
9	0.75	0.75	60.04

• Conclusions

The potential issue above, which is determined by Laplace's equation, amply illustrates how useful boundary element methods are. It is also evident that boundary element methods have a versatile and straightforward application. It's also evident that adding more parts and nodes produces better outcomes in line with expectations. It is possible to apply the same idea to other kinds of elliptic partial differential equations.

Competing Interests

The authors have stated that there are no conflicting interests.

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